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A variational approach to an unsymmetric water-wave scattering problem

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SUMMARY

A plane surface wave train on infinitely deep water is incident upon a pair of fixed thin vertical barriers, one of which is in the surface, the other submerged. The relation between the input and output amplitudes is obtained via a variational approximation for large barrier separations. It is shown that, within this approximation, infinite spectra of totally reflected and totally transmitted waves exist if the barriers overlap, but for non-overlapping barriers this is not the case.

1. Introduction

The Schwinger variational method is a long-established technique in the theory of waveguides, which of recent years has also been successfully applied to water-wave problems. The power of the method lies in its ability to provide simple and accurate approximations to the phase and amplitude of scattered waves in the far-field, without the necessity of calculating the near-field behaviour in detail. It was first applied in the context of water-waves by Miles [1] to the problem of diffraction by a finite step; subsequent applications include the work of Mei and Black [2] and Black, Mei and Bray [3] on scattering by surface and bottom obstacles, and that of Evans and Morris [4, 5] on vertical barriers. Numerical evidence presented by these authors and others, comparing results obtained by this method with those derived from such exact solutions as are available, supports the accuracy of the method. In particular, for problems possessing a certain type of symmetry there exist two complementary variational formulations which actually yield upper and lower bounds on the solution ([4, 5]).

All the papers cited above deal with problems which can be cast in the form of a single scalar integral equation; it is when this equation has a symmetric and positive- or negative-definite kernel that the bounds mentioned above may be obtained. The underlying theory, however, as given for example in Stakgold [6], chapter 8, holds for a much wider class of problems expressible in terms of linear operators on Hilbert spaces. In the present paper we demonstrate how the variational method may be made to cope in practice with an unsymmetric problem which reduces to the solution of a pair of simultaneous integral equations over different regions, or equivalently, of a two-dimensional vector integral equation with unsymmetric kernel. The extension to higher dimensions is trivial. In such a situation, of course, though we may still formulate the two "complementary" variational expressions, these, as pointed out by Stakgold [6], no longer furnish a maximum and a minimum principle. Instead we have simply two stationary principles, leading to two approximate solutions but giving no guarantee that these are bounds for the exact solution.

The problem to be considered here is also of some interest from a physical standpoint. A plane wave train is incident upon the configuration illustrated in Figure 1 of two parallel fixed vertical plane barriers, one extending from the free surface to a depth a, the other from a depth b down to infinity. If the barrier separation is $2w$, and the wave-number K, we can form three independent dimensionless ratios, b/a, w/a and Ka, on which the solution will depend. The question then arises whether there exist combinations of values of these ratios which result in either total transmission or total reflection of the wave energy. In his examination of the case of diffraction by two equal submerged barriers, Jarvis [7] found that total transmission, but not total reflection, could occur; for two equal surface-piercing barriers, however, Evans and Morris [5] found both totally transmitted and totally reflected wavelengths.

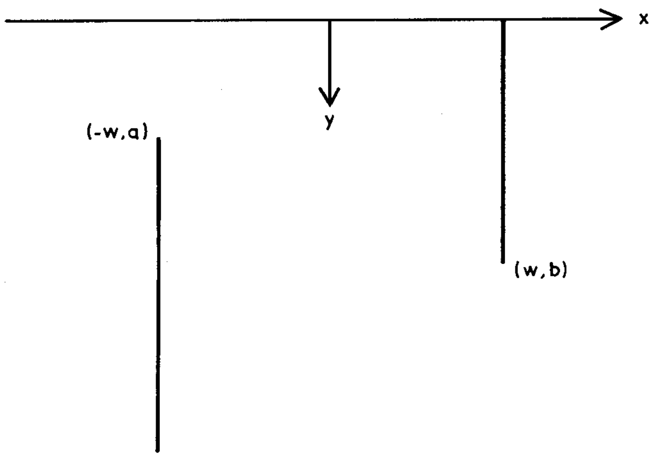

Figure 1. Barrier configuration.

A subsequent more detailed analysis of this latter problem by Newman [8] for the particular case when the barriers are close together revealed that the zeros of reflection and transmission are due to resonance effects in the oscillating column of fluid between the barriers. We might thus anticipate that in the present problem there would be a significant qualitative difference between the ranges $b/a>1$ (barriers overlapping) and $b/a\leqq 1$ (no overlap). It will be shown below that this is in fact the case, in that the overlap situation gives rise to an infinite sequence of zeros of both reflection and transmission, while in the non-overlap situation there is at most a finite set of zeros of each.

2. Statement of the problem and outline of method

Figure 1 illustrates the situation to be considered. The problem is strictly two-dimensional, so that the figure may be understood to be infinitely extended in the $\pm z$-directions. The y-axis is taken vertically downwards, the x-axis in the mean free surface; thus the fluid region is $y\geqq 0$. The lines $(-w, a)$ to $(-w, \infty)$ and $(w, 0)$ to (w, b) are occupied by thin rigid fixed barriers. The assumptions of the linearized, small-amplitude wave theory for an inviscid fluid in irrotational motion are made throughout.

Then there exists a velocity potential $\Phi(x, y, t)$ for the motion, which we assume can be expressed in terms of a complex potential $\phi(x, y)$ by

$$\Phi(x, y, t) = Re[\phi(x, y)e^{-i\sigma t}] .$$

The function ϕ then satisfies the system

$$\nabla^2\phi = 0 , \qquad y\geqq 0 , \tag{1}$$

$$K\phi + \frac{\partial\phi}{\partial y} = 0 , \qquad y=0 , \tag{2}$$

where $K=\sigma^2/g$,

$$\phi, \nabla\phi\rightarrow 0 , \quad \text{as } y\rightarrow\infty , \tag{3}$$

and

$$r\frac{\partial\phi}{\partial r}\rightarrow 0 , \quad \text{as } r\rightarrow 0 , \tag{4}$$

where $r^2=(x\pm w)^2+(y-{a \atop b})^2$.

This last condition expresses the requirement that there should be at worst an integrable singularity in the velocity at each of the barrier edges.

In addition, we must prescribe the behaviour of ϕ as $x\to\pm\infty$. Ultimately we shall be concerned with the reflection and transmission of a wave incident from $x=-\infty$, but for the time being we allow incident waves from either infinity, so that

$$\phi \sim \alpha_1 e^{iKx}+\beta_1 e^{-iKx}, \quad x\to-\infty, \tag{5}$$

$$\phi \sim \alpha_2 e^{-iKx}+\beta_2 e^{iKx}, \quad x\to+\infty. \tag{6}$$

We consider the potentials in the three regions $-\infty\leqq x<-w$, $-w<x<w$, and $w<x\leqq\infty$ separately, and write

$$\phi_1(x,y) = -(A_1 e^{iK(x+w)}+B_1 e^{-iK(x+w)})e^{-Ky} + \int_0^\infty \frac{S_1(u)(u\cos uy-K\sin uy)e^{u(x+w)}du}{u(u^2+K^2)}, \quad x<-w,$$

$$\phi_2(x,y) = (A_2 e^{iKx}+B_2 e^{-iKx})e^{-Ky} + \int_0^\infty \frac{(S_{21}(u)e^{ux}+S_{22}(u)e^{-ux})(u\cos uy-K\sin uy)du}{u(u^2+K^2)}, \quad -w<x<w,$$

$$\phi_3(x,y) = (A_3 e^{-iK(x-w)}+B_3 e^{iK(x-w)})e^{-Ky} + \int_0^\infty \frac{S_3(u)(u\cos uy-K\sin uy)e^{-u(x-w)}du}{u(u^2+K^2)}, \quad x>w.$$

Here we follow Havelock [9] and others in writing the potential as a sum of wavelike terms together with an integral representing the decaying part of the solution. Were we to consider finite-depth effects, the integrals would be replaced by sums over an infinite set of discrete eigenvalues u. It is easily seen that $\phi_i(x,y)$, $i=1,2,3$, satisfy (1) to (6), with complex input amplitudes

$$\alpha_1 = -A_1 e^{iKw}, \quad \alpha_2 = A_3 e^{iKw}, \tag{7}$$

and output amplitudes

$$\beta_1 = -B_1 e^{-iKw}, \quad \beta_2 = B_3 e^{-iKw}. \tag{8}$$

The object of the variational approach is to obtain the output amplitudes from a prescribed input without determining the functions $S_j(u)$ or the constants A_2, B_2. As mentioned in Section 1 above, there are two possible formulations for the variational expression; we may take as unknown functions either the velocities across $x=-w$, $x=w$, or the potential-differences (pressure) across these lines. The two methods would, for a symmetric problem, yield the complementary maximum and minimum principles. We give here the details only for the velocity form, and simply summarise the results for the pressure form, which goes through very similarly.

3. Velocity formulation

Define the intervals $g_1=(0,a)$ and $g_2=(b,\infty)$. Also define

$$\left(\frac{\partial\phi}{\partial x}\right)_{x=-w} = U_1(y) \text{ and } \left(\frac{\partial\phi}{\partial x}\right)_{x=w} = U_2(y),$$

so that $U_i(y)=0$, $y\notin g_i$. Then

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{x=-w} = U_1(y) = -iK(A_1-B_1)e^{-Ky} + \int_0^\infty \frac{S_1(u)(u\cos uy-K\sin uy)du}{u^2+K^2},$$

and

$$\left(\frac{\partial\phi_2}{\partial x}\right)_{x=w} = U_1(y)$$
$$= iK(A_2\,\mathrm{e}^{-iKw} - B_2\,\mathrm{e}^{iKw})\,\mathrm{e}^{-Ky}$$
$$+ \int_0^\infty \frac{(S_{21}(u)\mathrm{e}^{-uw} - S_{22}(u)\mathrm{e}^{uw})(u\cos uy - K\sin uy)\,du}{u^2+K^2},$$

Hence, by an inversion theorem due to Havelock [9],

$$-A_1 + B_1 = A_2\,\mathrm{e}^{-iKw} - B_2\,\mathrm{e}^{iKw} = -2i\int_0^\infty U_1(y)\,\mathrm{e}^{-Ky}\,dy, \tag{9}$$

and

$$S_1(u) = S_{21}(u)\mathrm{e}^{-uw} - S_{22}(u)\mathrm{e}^{uw} = \frac{2}{\pi}\int_0^\infty U_1(y)(u\cos uy - K\sin uy)\,dy. \tag{10}$$

Similarly by matching $\partial\phi_2/\partial x$ and $\partial\phi_3/\partial x$ at $x=w$ with $U_2(y)$, we obtain

$$B_3 - A_3 = A_2\,\mathrm{e}^{iKw} - B_2\,\mathrm{e}^{-iKw} = -2i\int_0^\infty U_2(y)\,\mathrm{e}^{-Ky}\,dy \tag{11}$$

and

$$-S_3(u) = S_{21}(u)\mathrm{e}^{uw} - S_{22}(u)\mathrm{e}^{-uw} = \frac{2}{\pi}\int_0^\infty U_2(y)(u\cos uy - K\sin uy)\,dy. \tag{12}$$

From (9) and (11) A_2 and B_2, and from (10) and (12) $S_{21}(u)$ and $S_{22}(u)$, may be determined:

$$A_2 = \tfrac{1}{2}i\,\mathrm{cosec}\,2Kw\,[\mathrm{e}^{iKw}(A_3 - B_3) - \mathrm{e}^{-iKw}(A_1 - B_1)], \tag{13}$$

$$B_2 = \tfrac{1}{2}i\,\mathrm{cosec}\,2Kw\,[\mathrm{e}^{-iKw}(A_3 - B_3) - \mathrm{e}^{iKw}(A_1 - B_1)], \tag{14}$$

$$S_{21}(u) = -\tfrac{1}{2}\,\mathrm{cosech}\,2uw\,[\mathrm{e}^{-uw}S_1(u) + \mathrm{e}^{uw}S_3(u)], \tag{15}$$

and

$$S_{22}(u) = -\tfrac{1}{2}\,\mathrm{cosech}\,2uw\,[\mathrm{e}^{uw}S_1(u) + \mathrm{e}^{-uw}S_3(u)]. \tag{16}$$

Now the pressure-difference is zero at $x=-w$ for $y\in g_1$, so that $\phi_1=\phi_2$, giving

$$\mathrm{e}^{-Ky}(A_1 + B_1 + A_2\,\mathrm{e}^{-iKw} + B_2\,\mathrm{e}^{iKw})$$
$$= \int_0^\infty \frac{(S_1(u) - S_{21}(u)\mathrm{e}^{-uw} - S_{22}(u)\mathrm{e}^{uw})(u\cos uy - K\sin uy)\,du}{u(u^2+K^2)},$$

for $y\in g_1$. Eliminating the $S_{ij}(u)$, A_2 and B_2 via (10) and (12) to (16), and re-arranging, we obtain the integral equation

$$i\,\mathrm{cosec}\,2Kw(A_3 - B_3 - A_1\,\mathrm{e}^{2iKw} + B_1\,\mathrm{e}^{-2iKw})\,\mathrm{e}^{-Ky}$$
$$= \frac{2}{\pi}\int_0^\infty [U_1(\eta)h(y,\eta) - U_2(\eta)g(y,\eta)]\,d\eta, \quad y\in g_1, \tag{17}$$

where the symmetric kernel functions are

$$g(y,\eta) = \int_0^\infty (u\cos uy - K\sin uy)(u\cos u\eta - K\sin u\eta)\,\mathrm{cosech}\,2uw/u(u^2+K^2)\,du \tag{18}$$

and

$$h(y,\eta) = \int_0^\infty (u\cos uy - K\sin uy)(u\cos u\eta - K\sin u\eta)\mathrm{e}^{2uw}\,\mathrm{cosech}\,2uw/u(u^2+K^2)\,du. \tag{19}$$

In the same way, by setting $\phi_2=\phi_3$ at $x=w$ for $y\in g_2$, and rearranging, we obtain the second integral equation

$$i \operatorname{cosec} 2Kw(A_1 - B_1 - A_3 e^{2iKw} + B_3 e^{-2iKw}) e^{-Ky} = \frac{2}{\pi} \int_0^\infty [U_2(\eta) h(y, \eta) - U_1(\eta) g(y, \eta)] \, d\eta \,, \quad y \in g_2 \,. \tag{20}$$

As they stand, (17) and (20) cannot be combined in a single vector integral equation, since they are valid for different ranges of values of y. However, they can be extended so that each is valid in $y \in (0, \infty)$, as follows.

Introduce the Heaviside unit functions $H_i(y)$ defined by

$$H_1(y) = H(a-y) = \begin{cases} 1, & y < a \\ 0, & y > a \end{cases} \quad \text{and} \quad H_2(y) = H(y-b) = \begin{cases} 1, & y > b \\ 0, & y < b \,. \end{cases}$$

Then, by virtue of the vanishing of $U_i(y)$ outside g_i, we have $U_i(y) H_i(y) \equiv U_i(y)$. Thus (17) may be written as

$$i \operatorname{cosec} 2Kw(A_3 - B_3 - A_1 e^{2iKw} + B_1 e^{-2iKw}) e^{-Ky} H_1(y) = \frac{2}{\pi} \int_0^\infty [U_1(\eta) H_1(\eta) h(y, \eta) - U_2(\eta) H_2(\eta) g(y, \eta)] \, d\eta \, H_1(y) \,, \tag{21}$$

and (20) as

$$i \operatorname{cosec} 2Kw(A_1 - B_1 - A_3 e^{2iKw} + B_3 e^{-2iKw}) e^{-Ky} H_2(y) = \frac{2}{\pi} \int_0^\infty [U_2(\eta) H_2(\eta) h(y, \eta) - U_1(\eta) H_1(\eta) g(y, \eta)] \, d\eta \, H_2(y) \,, \tag{22}$$

both equations now being valid for $y \in (0, \infty)$. (21) and (22) may now be combined in the form

$$(\boldsymbol{A} D + \boldsymbol{B} \bar{D}) H(y) e^{-Ky} = \frac{2}{\pi} \int_0^\infty \boldsymbol{U}(\eta) N(y, \eta) \, d\eta \,, \quad y \in (0, \infty) \,, \tag{23}$$

where $\boldsymbol{A} = (A_1, A_3)$, $\boldsymbol{B} = (B_1, B_3)$, $\boldsymbol{U}(y) = (U_1(y), U_2(y))$,

$$H(y) = \begin{pmatrix} H_1(y) & 0 \\ 0 & H_2(y) \end{pmatrix}, \quad D = \begin{pmatrix} -e^{2iKw} & 1 \\ 1 & -e^{2iKw} \end{pmatrix},$$

$$P(y, \eta) = \begin{pmatrix} h(y, \eta) & -g(y, \eta) \\ -g(y, \eta) & h(y, \eta) \end{pmatrix}, \quad \text{and} \quad N(y, \eta) = H(\eta) P(y, \eta) H(y) \,.$$

We next introduce a matrix $u(y)$ which is related to $\boldsymbol{U}$ by

$$\boldsymbol{U}(y) = (\boldsymbol{A} D + \boldsymbol{B} \bar{D}) u(y) \,, \tag{24}$$

so that (23) becomes

$$(\boldsymbol{A} D + \boldsymbol{B} \bar{D}) H(y) e^{-Ky} = \frac{2}{\pi} (\boldsymbol{A} D + \boldsymbol{B} \bar{D}) \int_0^\infty u(\eta) N(y, \eta) \, d\eta \,, \quad y \in (0, \infty) \,,$$

which will follow if $u(y)$ is a solution of

$$H(y) e^{-Ky} = \frac{2}{\pi} \int_0^\infty u(\eta) N(y, \eta) \, d\eta \,, \quad y \in (0, \infty) \,. \tag{25}$$

Note that although $u(y)$ is not uniquely defined by (24), a solution of (25), however obtained, will when inserted in (24) yield a function $\boldsymbol{U}(y)$ satisfying (23).

Postmultiplying (25) by $u'(y)$ (where $'$ denotes transpose) and integrating with respect to y over $(0, \infty)$ gives

$$\int_0^\infty H(y) u'(y) e^{-Ky} \, dy = \frac{2}{\pi} \int_0^\infty \int_0^\infty u(\eta) N(y, \eta) u'(y) \, d\eta \, dy \,,$$

or, in component form,

$$\int_0^\infty H_i(y) u_{ji}(y) e^{-Ky} \, dy = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sum_{k, m=1}^{2} u_{ik}(\eta) N_{km}(y, \eta) u_{jm}(y) \, d\eta \, dy \,. \tag{26}$$

To obtain the stationary form we must return to equations (9) and (11) for the input and output amplitudes, and observe that they may be written together as

$$A - B = 2i \int_0^\infty U(y) H(y) e^{-Ky} dy$$
$$= 2i(AD + B\bar{D}) \int_0^\infty u(y) H(y) e^{-Ky} dy , \tag{27}$$

on using (24).

We now introduce the scattering matrix* S by the relation

$$A - B = 2i(AD + B\bar{D})S \tag{28}$$

Then (27) becomes

$$2i(AD + B\bar{D})S = 2i(AD + B\bar{D}) \int_0^\infty u(y) H(y) e^{-Ky} dy ,$$

which will hold for an S such that

$$S = \int_0^\infty u(y) H(y) e^{-Ky} dy ,$$

i.e.

$$S_{ji} = \int_0^\infty u_{ji}(y) H_i(y) e^{-Ky} dy . \tag{29}$$

But (29) can be re-written as

$$S_{ji} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sum_{k,m=1}^{2} u_{ik}(\eta) N_{km}(y, \eta) u_{jm}(y) d\eta\, dy , \tag{30}$$

whence it is immediately clear, since $N'(y, \eta) = N(\eta, y)$, that S is a real symmetric matrix.

Combining (29) and (30) yields the relation

$$S_{ij} = \frac{\int_0^\infty u_{ij}(y) H_j(y) e^{-Ky} dy \int_0^\infty u_{ji}(y) H_i(y) e^{-Ky} dy}{\frac{2}{\pi} \int_0^\infty \int_0^\infty \sum_{k,m=1}^{2} u_{ik}(\eta) N_{km}(y, \eta) u_{jm}(y) d\eta\, dy} . \tag{31}$$

It may be shown in the usual way, again using the fact that $N'(y, \eta) = N(\eta, y)$, that S is stationary with respect to small variations of $u(y)$ about the solution of (25).

Once S has been determined by the use of a suitable approximation in (31), the relation between input and output amplitudes is very simply found. Re-arranging (28),

$$B = AX , \tag{32}$$

where

$$X = (I - 2iDS)(I + 2i\bar{D}S)^{-1} ,$$

whence B may be obtained when A is prescribed. The energy conservation law in the form $|A|^2 = |B|^2$, and Kreisel's [10] symmetry relations for waves incident from either infinity, may be proved using (32).

4. The choice of approximating functions

The success of the variational method depends on the choice of functions to approximate $u_{ij}(y)$. Following Evans and Morris [4, 5] we make use of the known exact solutions for a single surface barrier and a single submerged barrier, due to Ursell [11] and Dean [12] respectively. Then we may expect that the approximation will be good for large barrier separations, when each barrier does not "feel" the presence of the other to any great extent.

Since the functions $u_{ij}(y)$ are associated with the velocities $U_j(y)$ via (24), we set $u_{ij}(y) = a_{ij} f_j(y)$, where a_{ij} are constants and $f_j(y)$ will be derived below from the appropriate exact solutions. Then (31) becomes

* This is not quite the "scattering matrix" in the sense that the term is used in electromagnetic wave theory; it is, however, in line with the use of the term by Miles [1].

$$S_{ij}=\frac{a_{ij}a_{ji}\int_{g_i}f_i(y)e^{-Ky}dy\int_{g_j}f_j(y)e^{-Ky}dy}{\frac{2}{\pi}\sum_{k,m=1}^{2}a_{ik}a_{jm}\int_{g_k}\int_{g_m}f_k(\eta)P_{km}(y,\eta)f_m(y)d\eta dy}\,.$$

But, S_{ij} being stationary, we have $\partial S_{ij}/\partial a_{rs}=0\,(i,j,r,s=1,2)$, whence arises a set of homogeneous equations for the coefficients a_{rs}. Consistency conditions applied to these equations obviate the need to determine the a_{rs}, giving directly

$$S_{11}=\frac{\pi l_1^2 l_{11}}{2|l|},\quad S_{12}=\frac{\pi l_1 l_2 l_{12}}{2|l|},\quad S_{22}=\frac{\pi l_2^2 l_{22}}{2|l|},$$

where

$$l_i=\int_{g_i}f_i(y)e^{-Ky}dy\,,\quad l_{11}=\int_{g_1}\int_{g_2}f_2(\eta)P_{22}(y,\eta)f_2(y)d\eta dy\,,$$

$$l_{12}=\int_{g_1}\int_{g_2}f_1(\eta)P_{12}(y,\eta)f_2(y)d\eta dy\,,$$

and

$$l_{22}=\int_{g_1}\int_{g_1}f_1(\eta)P_{11}(y,\eta)f_1(y)d\eta dy\,.$$

On replacing P_{rs} by the explicit forms of the kernels (18) and (19) we find that there are four basic integrals to be evaluated, namely

$$\int_0^a f_1(y)e^{-Ky}dy\,,\quad \int_b^\infty f_2(y)e^{-Ky}dy\,,$$

$$\int_0^a f_1(y)(u\cos uy-K\sin uy)dy\,,\ \text{and}\ \int_b^\infty f_2(y)(u\cos uy-K\sin uy)dy\,.$$

The appropriate functions in the exact single barrier solutions are actually transformations of f_1 and f_2; thus, for example, if we define

$$\psi_1(y)=K\int_y^a f_1(u)du-f_1(y)\,,$$

then the corresponding form given by Dean [12] for the single submerged barrier is $\psi_1(y)=(a^2-y^2)^{-\frac{1}{2}}$ (where we have set an arbitrary constant equal to one without loss of generality, since the expression for S_{ij} is scale-invariant). Integration by parts now shows that

$$\int_0^a f_1(y)e^{-Ky}dy=-\int_0^a\psi_1(y)\cosh Ky\,dy=-\tfrac{1}{2}\pi I_0(aK)$$

and

$$\int_0^a f_1(y)(u\cos uy-K\sin uy)dy=-u\int_0^a\psi_1(y)\cos uy\,dy$$
$$=-\tfrac{1}{2}\pi J_0(au)\,,$$

with the usual Bessel function notation.

Similarly if $\psi_2(y)=f_2(y)+K\int_b^y f_2(u)du$, then ψ_2 has been given by Ursell [11] for a single surface barrier as $y(y^2-b^2)^{-\frac{1}{2}}$, whence

$$\int_b^\infty f_2(y)e^{-Ky}dy=\frac{1}{2}\int_b^\infty\psi_2(y)e^{-Ky}dy=\tfrac{1}{2}bK_1(Kb)$$

and

$$\int_b^\infty f_2(y)(u\cos uy-K\sin uy)dy=u\int_b^\infty[\psi_2(y)-\psi_2(\infty)]\cos uy\,dy$$
$$-\psi_2(\infty)\sin ub=-\tfrac{1}{2}\pi ubJ_1(ub)\,.$$

Altogether, then, $l_1=-\frac{1}{2}\pi I_0(aK)$, $l_2=\frac{1}{2}bK_1(Kb)$,

$$l_{11} = \frac{\pi^2 b}{4} \int_0^\infty \frac{u J_1^2(bu) e^{2uw} \operatorname{cosech} 2uw \, du}{u^2+K^2},$$

$$l_{22} = \frac{\pi^2}{4} \int_0^\infty \frac{u J_0^2(au) e^{2uw} \operatorname{cosech} 2uw \, du}{u^2+K^2},$$

$$l_{12} = -\frac{\pi^2 b}{4} \int_0^\infty \frac{u J_0(au) J_1(bu) \operatorname{cosech} 2uw \, du}{u^2+K^2}.$$

5. Pressure formulation

If, instead of taking the velocities $U_i(y)$ as our unknown functions, we take the pressure-differences $V_1(y)=\phi_2(-w, y)-\phi_1(-w, y)$ and $V_2(y)=\phi_3(w, y)-\phi_2(w, y)$, we obtain a different variational principle leading to the result $\boldsymbol{B}=A\,Y$, where

$$Y = (2K^2 iT+D)(2K^2 iT-\bar{D})^{-1},$$

$$T_{11} = \frac{\pi L_1^2 L}{|L|}, \quad T_{12} = \frac{\pi L_1 L_2 L_{12}}{|L|}, \quad T_{22} = \frac{L_2^2 L_{22}}{|L|},$$

$$L_1 = \frac{1}{2K} K_0(aK), \quad L_2 = -\frac{\pi b}{2K} I_1(bK),$$

$$L_{11} = \frac{\pi^2 b^2}{4} I_1(bK) K_1(bK), \quad L_{22} = \frac{\pi^2}{4} I_0(aK) K_0(aK),$$

and

$$L_{12} = \frac{\pi^2 b}{4} \int_0^\infty \frac{u J_0(au) J_1(bu) e^{-2uw} du}{u^2+K^2}.$$

6. The reflection and transmission coefficients

If $A=(1, 0)$, we have an incident wave from the left only, with complex amplitude $-e^{iKw}$, so that the complex reflection and transmission coefficients are $r=B_1 e^{-2iKw}$ and $t=-B_3 e^{-2iKw}$ respectively.

But, in the velocity form, $\boldsymbol{B}=A\,X$, from (32), whence if $A=(1, 0)$, then $B_1=X_{11}$ and $B_3=X_{12}$. Thus $r=X_{11}e^{-2iKw}$ and $t=-X_{12}e^{-2iKw}$. Similarly from the pressure form, $r=Y_{11}e^{-2iKw}$ and $t=-Y_{12}e^{-2iKw}$. After some manipulation, these forms lead to

$$t = \frac{-4i(2|S|-S_{12}\sin 2Kw)e^{-2iKw}}{\sin 2Kw+4S_{12}-2e^{-2iKw}(S_{11}+S_{22})-8ie^{-2iKw}|S|} \tag{33}$$

$$= \frac{-i(1+2T_{12}K^2\sin 2Kw)e^{-2iKw}}{ie^{-2iKw}+2K^2T_{12}+K^2e^{-2iKw}(T_{11}+T_{22})-2K^4\sin 2Kw|T|}, \tag{34}$$

where $|T|$ denotes the determinant of T. Thus $|t|$ may be calculated; and similar formulae may be obtained leading to r.

7. Discussion of the solution

We first verify that the exact single barrier solutions are recovered if either barrier vanishes. If $b\to 0$ for fixed a, then using the pressure form (34) we have $T_{12}\to 0$, $T_{22}\to 0$, and $T_{11}\to K_0(aK)/\pi K^2 I_0(aK)$, so that $|t|\to\pi I_0(aK)/[\pi^2 I_0^2(aK)+K_0^2(aK)]^{\frac{1}{2}}$, in agreement with Dean's [12] result for a single barrier submerged to depth a.

If $a\to\infty$ with b fixed, then $T_{11}\to 0$, $T_{12}\to 0$, and $T_{22}\to\pi I_1(bK)/K_1(bK)$, giving $|t|\to K_1(bK)/[\pi^2 I_1^2(bK)+K_1^2(bK)]^{\frac{1}{2}}$; this agrees with the result obtained by Ursell [11] for a single surface-piercing barrier of length b. Exactly the same agreement follows if the velocity form (33) is used.

The asymptotic result of Newman [13] for long obstacles with horizontal mid-sections may be applied to the present problem in the limit as $w \to \infty$. His equation (3.1) then becomes

$$|t| \sim \frac{|t_b t_s|}{|1 - r_b r_s e^{4iKw}|}, \text{ as } w \to \infty ,$$

where suffices s, b denote values of the coefficients for the single surface and bottom obstacles respectively.

Using the pressure form again, we find that $L_{12} \to 0$ as $w \to \infty$, so that $T_{11} \to K_0(aK)/\pi K^2 I_0(aK)$, $T_{12} \to 0$, and $T_{22} \to \pi I_1(bK)/K^2 K_1(bK)$. However, from the results for single barriers cited above we may write $T_{11} \to ir_b/K^2 t_b$, $T_{22} \to ir_s/K^2 t_s$, from which it follows that

$$t \sim \frac{-t_b t_s}{1 - r_s r_b e^{4iKw}}, \text{ as } w \to \infty ; \tag{35}$$

thus $|t|$ is identical with the value given by Newman [13]. Moreover, in this limit the velocity approximation gives $S_{11} \to \pi I_0(aK)/4K_0(aK)$, $S_{12} \to 0$, $S_{22} \to \pi K_1(bK)/4I_1(bK)$, whence it is easily verified that the value (35) for t is again obtained. In other words, the two approximations converge in the limit $w \to \infty$, supporting our earlier statement that the choice of trial functions gives better results for larger barrier separations.

We next examine the asymptotic behaviour as $K \to \infty$, the other parameters remaining fixed. It is found that $T_{11} = O(K^{-2} e^{-2aK})$, $T_{22} = O(K^{-2} e^{2bK})$, and $T_{12} = O(K^{-3} e^{K(b-a)})$, whence $|t| = O(e^{-2bK})$, unless $b = 0$, when $|t| \to 1$. Thus for any finite value of b, however small, the amplitude of the transmitted wave tends to zero for very short wavelengths; if the surface barrier vanishes entirely, however, then short wavelengths are totally transmitted. This is physically reasonable, since for arbitrarily short waves only the situation actually at the surface is "felt" by the wave, so the difference between $b = 0$ and b small but finite becomes crucial.

Finally we turn to the question of the existence of totally transmitted or reflected wavelengths. Again we work with the pressure approximation (34), though the velocity form yields identical information. Equation (34) reveals that $t = 0$ if

$$1 + 2K^2 T_{12} \sin 2Kw = 0 ,$$

i.e.

$$\sin 2Kw = -1/2K^2 T_{12}(a, b, K, w) ,$$

or

$$\sin 2\lambda\mu = -a^2/2\lambda^2 T_{12}(\lambda, \mu, \nu) , \tag{36}$$

defining dimensionless variables $\lambda = Ka$, $\mu = w/a$, $\nu = b/a$. This equation will have solutions only if the right-hand side (r.h.s.) has modulus less than 1 for some combination of the parameters. However, we saw above that $T_{12} = O(K^{-3} e^{K(b-a)})$ as $K \to \infty$, so $T_{12}/a^2 = O(\lambda^{-3} e^{\lambda(\nu-1)})$, and the r.h.s. of (36) is therefore $O(\lambda e^{\lambda(1-\nu)})$, $\lambda \to \infty$.

So if $\nu > 1$, the r.h.s. of (36) tends to zero as $\lambda \to \infty$, and there exists some value λ_0 such that when $\lambda > \lambda_0$, the r.h.s. is less than 1. Thus for any $\nu > 1$ (i.e. $b > a$, which corresponds to the case where the barriers overlap) there are an infinite number of roots $\lambda(\mu)$ of equation (36) for each value of μ, giving an infinite spectrum of totally reflected wavelengths. If, however, $\nu \leqq 1$, then the r.h.s. of (36) becomes infinite as $\lambda \to \infty$, so that in this case (36) can have, at most, a finite set of roots $\lambda(\mu)$. A similar analysis shows that an analogous result holds for totally transmitted wavelengths.

8. Conclusion

It has been shown that according to the variational approximation adopted here, the occurrence of total transmission or reflection depends crucially on the value of the parameter b/a which measures the amount of overlap of the barriers. Since we do not in this case have bounds for the exact solution, we cannot deduce that the above results will carry over identically for the exact values of t and r; but in view of the convergence of the two approximations as $w \to \infty$

pointed out in Section 7, they may be regarded as highly suggestive. For a more detailed examination of the solution for smaller values of w, particularly in the vicinity of $b/a = 1$ where some kind of singular behaviour may be anticipated, an alternative approach must be used, perhaps along similar lines to that of Newman [8]. It is hoped to pursue this question in the future.

The present method may easily be extended to deal with obliquely incident waves; the term u in the denominator and the exponential of the integrated term in $\phi_i(x, y)$ is replaced by $(u^2 + K^2 \sin^2 \alpha)^{\frac{1}{2}}$, where α is the angle between the wave-fronts and the barriers. This results in a modified form of the matrices S, T, while the qualitative properties discussed in Section 7 remain unaltered.

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